Relativistic and Nonrelativistic Descriptions of Electron Energy Levels in a Static Magnetic Field

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Abstract

The physical consequences of the relativistic and nonrelativistic approaches to describe the energy levels of electrons which propagate in a static homogeneous magnetic field are considered. It is shown that for a given strength of the magnetic field, the quantized energy levels of the electrons calculated by nonrelativistic and relativistic equations differ substantially, up to few orders of magnitude for a magnetic field of about 1 Tesla. Experimental verification to resolve the discrepancy would be very welcome.

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1 Introduction

The existence of quantized transverse energy levels of charged particles which propagate in a static homogeneous magnetic field was predicted by the nonrelativistic Schrödinger equation [1] and the relativistic Klein-Gordon equation for a scalar particle [2, 3]. Later, analogous expressions for transverse energy levels of electrons were found within the Dirac equation, see e.g. [4]. For more details we refer to [5–8]. Furthermore, exact solutions were also derived for somewhat more complicated cases when the electron pass through a combined static electric and magnetic field (Volkov type solutions, see also [9,10]), and recently, this study was continued by finding the solution of the Dirac equation for a superposition of a static homogeneous magnetic and electric field including the anomalous magnetic moment of the electron [11]. It was also shown in ref. [11] that by accounting for the electron anomalous magnetic moment the spin degeneracy of energy levels was removed. So, according to all these results, an electron in a static magnetic field gyrates about, and moves along, the field lines and possesses quasi-atomic bound states with energy levels related to its gyrating motion in the plane normal to the velocity vector. In the literature, these quasi-discrete resonance states were discussed in connection with the motion of charged particles in an external magnetic field and accelerator physics, see [12–15] for example [1].

The analytical solutions of the corresponding equations make clear what kind of physical effects may appear during the transition from the nonrelativistic to the relativistic formalism. The example of the Coulomb potential, which we mention in what follows, shows a nontrivial physical sequence when passing from one case to the other.

In this note we would like to draw the attention to the fact that the nonrelativistic and relativistic approaches, based on the Schrödinger, respectively, the Dirac equation, give different analytical expressions for transverse energy levels of electrons in a static magnetic field. The solutions predict different dependencies of these levels on the magnetic field, so that for a given field strength, rather different values for transverse energies are expected. In order to check the validity of the different theoretical predictions it would be very welcome to measure the energy of photons emitted by electrons when transitions from higher to lower orbits occur in an external magnetic field.

In Section 2 we discuss some basic formulas which define the energy levels of an electron which traverses a static and uniform magnetic field. Section 3 presents the numerical values for transverse energy levels of electrons assuming a relatively strong magnetic field of 1 Tesla. Section 4 summarizes the discussion and proposes an experimental verification to decide which of the two theoretical concepts based on either the Dirac or Schrödinger equation is realized.

Throughout this paper, the Gaussian system of units will be used.

[1] The text books [6–8] include a rather complete review of possible applications starting from acceleration applications up to cosmology issues.

2 Energy levels and wave function of electrons in a static magnetic field; solutions of the Schrödinger and Dirac equations

The energy spectrum obtained from the Schrödinger equation for electrons with spin $\frac{1}{2}$ which are gyrating around the field lines of a static homogeneous magnetic field , ($\vec{H} = rot\vec{A}$) with, according to ref. [1], the following choice of the 4-vector of the electromagnetic potential A_μ: $A_0 = A_x = A_z = 0$, $A_y = Hx$ [2], may be written as the sum of the longitudinal and transverse components [16]

$$E^{nonrel} = E_z^{nonrel} + E_{T,\lambda}^{nonrel}(n) \ , \tag{1}$$

where

$$E_z^{nonrel} = \frac{p_z^2}{2m_e} \tag{2}$$

and

$$E_{T,\lambda}^{nonrel}(n) = \hbar(\frac{eH}{2m_e c})(2n+1+2\lambda) = (\mu_B^e H)(2n+1+2\lambda) \ , \tag{3}$$

the energy of the electron motion in transverse direction [3]. The transverse energy depends on the strength of the magnetic field H and on the electron spin projection λ onto the z-, i.e. the electron, direction. It possesses quantized values labeled by the main quantum number n ($n = 0, 1, 2, ...$). In eq.(3), m_e and e denote the electron mass, respectively, its charge, and $\frac{eH}{m_e c} = \omega_c$ is the cyclotron frequency. We employed here the definition of the Bohr magneton of an electron, $\mu_B^e = \frac{e\hbar}{2m_e c}$.

The energy levels, defined by the relativistic Dirac equation for electrons in the same magnetic field [4], are connected to the fourth component of its 4-momentum vector, $P_\mu = (p^0, p^x, p^y, p^z)$, and are given as

$$c^2(p^0)^2 \equiv E_\lambda^2(n, p_z) = E_z^2 + E_{T,\lambda}^2(n) \ . \tag{4}$$

The first term

$$E_z^2 = m_e^2 c^4 + p_z^2 c^2 \tag{5}$$

is the square of the relativistic energy of a free electron moving along the z-axis, whereas the second term defines the square of the relativistic electron transverse energy [4]:

$$\begin{aligned} E_{T,\lambda}^2(n) &= (m_e c^2)\hbar(\frac{eH}{m_e c})(2n+1+2\lambda) = (m_e c^2)\hbar\omega_c(2n+1+2\lambda) \\ &= 2m_e c^2(\mu_B^e H)(2n+1+2\lambda) \ . \end{aligned} \tag{6}$$

Comparing (3) and (6) the last equation may be expressed as

$$E_{T,\lambda}^2(n) = 2mc^2 \cdot E_{T,\lambda}^{nonrel}(n) \ . \tag{7}$$

Note that for the ground state (with $n = 0$) and spin projection $\lambda = -\frac{1}{2}$, the electron transverse energy is equal to zero in both the nonrelativistic and relativistic approaches:

$$E_{T,\lambda=-\frac{1}{2}}^2(n=0) \equiv E_{T,\lambda=-\frac{1}{2}}^2(0) = 0 \ , \tag{8}$$

[2] Thus, $\vec{A} = Hx\vec{e_y}$, with $\vec{e_y}$ the unit vector along the y-axis, and $\vec{H} = (0, 0, H_z)$, with $H_z = H$, and the electron momentum along the z-axis.

[3] Here $\hbar$ is the value of the Plank constant h divided by 2π, i.e. $\hbar = 1.054571586(82)10^{-34}$ J s.

[4] The solution of the Klein-Gordon equation [2,3] is obtained by omitting the term 2λ in eq.(3).

whereas the $n = 0$ level with spin projection $\lambda = +\frac{1}{2}$ has, according to (6), a non-zero transverse energy squared of

$$E^2_{T,\lambda=+\frac{1}{2}}(n = 0) = 2\hbar\omega_c(m_e c^2) = 2\ (2m_e c^2)(\mu^e_B H)\ . \tag{9}$$

So, we realize that *for the ground state with* $\lambda = -\frac{1}{2}$, *the relativistic expression for the total energy of an electron in a static magnetic field* coincides with *the energy of a free electron*

$$E_{\lambda=-\frac{1}{2}}(n = 0, p_z) = E_z = \sqrt{m_e^2 c^4 + p_z^2 c^2}\ , \tag{10}$$

which is however not the case for spin projection $\lambda = +\frac{1}{2}$. One also notices from eqs.(6) and (9) that the state with the quantum numbers $n = 0$ and $\lambda = +\frac{1}{2}$ has the same transverse energy as the state with $n = 1$ and $\lambda = -\frac{1}{2}$, i.e.

$$E^2_{T,\lambda=-\frac{1}{2}}(n = 1) = E^2_{T,\lambda=+\frac{1}{2}}(n = 0) = 2\hbar\omega_c(m_e c^2)\ = 2\ (2m_e c^2)(\mu^e_B H)\ . \tag{11}$$

From eq.(6) one derives for the difference $\Delta E^2_{T,\lambda}(n+k|n)$ of the square of two transverse energy levels $E^2_{T,\lambda}$, labeled as $n + k$ $(k = 1, 2, ...)$ and n, and identical spin projections λ, i.e. for the non-spinflip case, the following expression

$$\begin{aligned}\Delta E^2_{T,\lambda}(n + k|n) &\equiv E^2_{T,\lambda}(n + k) - E^2_{T,\lambda}(n) = 2k\hbar\omega_c(m_e c^2)\\ &= 4k(\mu^e_B H)(mc^2) = 2ec\hbar kH = k\Delta E^2_{T,\lambda=-1/2}(1|0)\ .\end{aligned} \tag{12}$$

The energy eigenvalues of eq.(4) in the nonrelativistic limit might be expanded to [6]

$$\begin{aligned}E_\lambda(n, p_z) &= \sqrt{E_z^2 + E^2_{T,\lambda}(n)} = \sqrt{m_e^2 c^4 + p_z^2 c^2 + \hbar(\frac{eH}{m_e c})(m_e c^2)(2n + 1 + 2\lambda)}\\ &\approx m_e c^2 + (\frac{p_z^2}{2m_e}) + (\mu_B H)(2n + 1 + 2\lambda) = m_e c^2 + E^{nonrel}\ .\end{aligned} \tag{13}$$

This equation clearly reveals the relationship between the relativistic energy $E_\lambda(n, p_z)$ to the nonrelativistic energy E^{nonrel} or vice versa.

3 Comparison of numerical values of transverse energy levels from the Schrödinger and Dirac equations

In the nonrelativistic Schrödinger case the transverse energy is, according to (3), proportional to the strength of the magnetic field H

$$E^{nonrel}_{T,\lambda}(n) \sim (\mu^e_B H)\ , \tag{14}$$

whereas in the relativistic case the transverse energy is, according to (6), proportional to the square root of H

$$E_{T,\lambda}(n) = \sqrt{2(m_e c^2)E^{nonrel}_{T,\lambda}(n)} \sim \sqrt{2(m_e c^2)(\mu^e_B H)}\ . \tag{15}$$

Obviously, there is a distinct different behavior of both solutions with respect to the magnetic field, which is expected to be more pronounced at larger magnetic field strengths.

For a numerical illustration, let us consider the case for a field of 1 Tesla. Utilizing the Bohr magneton of an electron (in Gauss units) [5]

$$\mu_B^e = \frac{e\hbar}{2m_e c} = 0.927\ 10^{-20}\ erg\ Gauss^{-1} = 5.788\ 10^{-9}\ eV\ Gauss^{-1} = \\ = 5.788\ 10^{-15}\ MeV\ Gauss^{-1} = 5.788\ 10^{-11}\ MeV\ T^{-1} \quad (16)$$

and eq.(3), the following value for the nonrelativistic transverse energy of the first exited state is obtianed

$$E^{nonrel}_{T,\lambda=-\frac{1}{2}}(n=1)_{H=1T} = 2\mu_B^e H = 2 \cdot 5.788\ 10^{-9}\ eV\ Gauss^{-1}\ 10^4\ Gauss \approx 1.158\ 10^{-4}\ eV. \quad (17)$$

If this number for $E^{nonrel}_{T,\lambda=-\frac{1}{2}}(n=1)_{H=1T}$ is introduced into (15), the following value for the relativistic transverse energy of an electron being in the same magnetic field with identical quantum numbers n and λ is found

$$E_{T,\lambda=-\frac{1}{2}}(n=1)_{H=1T} \approx 10.87\ eV\ . \quad (18)$$

Comparing the numbers in (17) and (18), one notices that for the magnetic field strength of 1 Tesla the relativistic Dirac equation provides for the first radial excitation a transverse energy *five orders of magnitude* higher than the nonrelativistic Schrödinger equation. Technically, this mismatch can be understood from formula (15) because a) the square root of H enlarges $E_{T,\lambda}$ by about two orders of magnitude and b) the second multiplication factor, $\sqrt{2(m_e c^2)}$ (with $m_e c^2 = 0.511\ MeV =\ 0.511 \cdot 10^6\ eV$), increases the energy level by additional three orders of magnitude. The substantial difference derived for a magnetic field of 1 Tesla, being expected to grow with increasing magnetic field strength, was to our knowledge never discussed in the literature so far.

In this connection it is of interest to recall that the situation with the physical interpretation of the exact solutions of the Dirac equation is not so definite in cases of strong electric fields. For example, it is well known that the solution for the energy levels of the Dirac equation using the Coulomb potential, $V(r) = -\frac{Ze^2}{r}$, where the charge factor Z defines the strength of the electric field, can be written as

$$E^{Coul.rel}_{n,j} = mc^2 (1 + \frac{(\alpha Z)^2}{(n - (j+1/2) + \sqrt{(j+1/2)^2 - (\alpha Z)^2}\ )^2})^{-1/2}\ , \quad (19)$$

with $\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$ and $n = j + 1/2 + k = 1, 2, ...$ as the main quantum number, and $j = 1/2$ for $l = 0$ or $j = l \pm 1/2$ if $l \neq 0$. Eq.(19) has however a restricted range of physical validity. For instance, for the smallest value $j = 1/2$ the expression under the square root in the dominator becomes negative and leads to unphysical solutions if $Z > Z_{cr}$, with $Z_{cr} = 137$ as the critical value for the charge factor.

[5] $erg = 0.624 \cdot 10^{12}$ eV, 10^4 Gauss = 1 Tesla.

In contrast to the relativistic expression (19), the analogous formula for the bound state energy levels within the Schrödinger equation

$$E_{n,j}^{Coul.nonrel} = -\frac{R\hbar Z^2}{n^2} \ , \tag{20}$$

with the Rydberg constant $R = me^4/2\hbar^3$ and $n = 1, 2, ...$, is valid for all Z values. To resolve experimentally the discrepancy between the relativistic solution (19) and the nonrelativistic expression (20), in particular for large values of Z, is challenging since pointlike charges with $Z \geq 137$ do not exist in nature [6].

Unlike the example of the Coulomb potential, the solution of the Dirac equation (6) for transverse energy levels of electrons within in a magnetic field is valid for any strength of the field H and, thereby, the experimental verification of the predictions (3), respectively, (6) is possible even at very high field strengths [7].

Such a task may be performed by passing an electron beam through a static homogeneous field of about 0.1-10 Tesla. Electrons of up to 100 MeV (which are considerd to be relativistic due to the smallness of their mass) may occupy some quasi-stable quantized energy levels $E_{T,\lambda}(n)$. For large life times of these levels, see [12] and [17], transitions of excited electrons to the ground state (10) are limited during passing through the H-field and registration of emitted photons should be performed sufficiently downstream of the magnet.

Also, "stimulation" of beam electrons by absorption of laser light inside a magnet leads to "exited" states of electrons which may be followed by emission of γ-rays within or behind the magnet depending on the lifetime of the excited states. Details of such an experiment should however be considered as soon as its realization becomes appropriate.

4 Summary

Numerical values for the transverse energy of electrons which propagate in a static homogeneous magnetic field were calculated using the relativistic (Dirac) and the nonrelativistic (Schrödinger) equations. Employing a magnetic field of 1 Tesla and non-spinflip transitions from orbits with $n = 1$ to $n = 0$, as an example, a difference of five orders of magnitude between the relativistic and nonrelativistic concepts for the electrons' transverse energy was evaluated. In other words, electrons traversing a magnetic field of 1 Tesla radiate photons being about 10^5 times more energetic in the relativistic than in the nonrelativistic case, for non-spinflip transitions from $n = 1$ to $n = 0$ orbits.

Finally, we believe that experimental verification of the predictions from either the relativistic or the nonrelativistic equation on quasi-atomic quantized energy levels of electrons traversing a strong static magnetic field would be very desirable. Such measurements can be performed by studying, for example, Compton scattering of laser light with electrons when both beams move parallel along the magnetic field lines. Registration of radiated photons, caused by electron transitions from higher to lower orbits, should resolve the difference between the relativistic and

[6] See, e.g. [4] - [8] for more discussions of this problem in connection of heavy ion cases.
[7] We have not yet found any results of such a study in the literature.

nonrelativistic predictions and should also provide a good test of how to add the interaction terms to the Dirac equation.

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